Integration of Discontinuous Expressions Arising in Beam Theory

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A technique of integrating discontinuous expressions arising in beam equations has, until recently, received little attention for a method as useful and simple to apply Recent mentions of this method in the literature and texts reference various origins for it This note traces the development of the "Clebsch Method" from its first known mention over a century ago through its development by many contributors to its present form

THERE exists a technique and formalism for differentiating and integrating discontinuous expressions arising in beam theory as though they were continuous expressions The cumbersome task of evaluating two integration constants in the deflection equation for each internal continuity condition at each discontinuity in the M/EI expression is circumvented; instead, one evaluates only two integration constants to satisfy the two external boundary conditions Recent presentations of this method in texts[1 2] and in the literature[3 4] give hope that it may receive the pre-eminence befitting such a powerful and simple tool This note traces the origin of this concept back to 1862 and reviews its subsequent development

One may use modern operational methods to determine the elastica of a particular beam and avoid the inconvenience of discontinuities in the M/EI expression which arise in classical solutions by integration However, this is a tool not generally employed by engineers confronted with such problems and not familiar to the beginning student in mechanics In 1862, a time when Heaviside was first formulating operational calculus, Clebsch[5] published a treatise on elasticity in which he suggested a method of solution of beam problems which today might be derived by the more sophisticated operational mathematics However, his arguments were based on observation and common sense

Clebsch examined the moment equations of a simply supported, uniform beam such as shown in Fig 1, i e ,

$$M = R_1 x \qquad 0 \leq x \leq a_1 \quad \text{(1a)}$$

$$M = R_1 x - P_1(x - a_1) \qquad a_1 \leq x \leq a_2 \quad \text{(1b)}$$

$$M = R_1 x - P_1(x - a_1) - P_2(x - a_2) \qquad a_2 \leq x \leq L \quad \text{(1c)}$$

He observed that, when the moment equations are written in the form of Eqs (1), each successive equation contains the terms of the previous equation plus one additional term arising from the discontinuity that necessitated writing a new equation Hence, if the term in parentheses $(x - a_i)$ is defined to be zero for $x < a_i$ and treated in the normal way for $x \geq a_i$, all three of Eqs (1) are contained in Eq (1c)

Equation (1c) is integrated to obtain the slope equation by substituting $u = x - a_i$ and $du = dx$ This results in

$$EI\theta = \int M dx = \tfrac{1}{2}R_1 x^2 - \tfrac{1}{2}P_1(x - a_1)^2 - \tfrac{1}{2}P_2(x - a_2)^2 + C_1 \quad (2)$$

which, upon recalling the special definition of the quantity $(x - a_i)$ just imposed, can be seen to satisfy the necessary internal continuity conditions A second integration following the same rules gives the equation of the elastica:

$$EIy = \tfrac{1}{6}R_1 x^3 - \tfrac{1}{6}P_1(x - a_1)^3 - \tfrac{1}{6}P_2(x - a_2)^3 + C_1 x + C_2 \quad (3)$$

The constants are evaluated to meet two external conditions A few quick checks, keeping in mind the special definitions implied by Clebsch, will assure one that these equations are in fact the same solution obtained by classical integration of the individual moment equations within their regions of continuity

To illustrate the advantages of his method, Clebsch formulated the necessary relationships to solve for the indeterminate reactions of a uniform beam on N equally spaced supports With no further consideration of a formalism for his technique, he moved on to more exciting topics in elasticity However, he had established a technique, although almost trivial to a man of his insight, by which the elastica could be determined in a simple, straightforward manner if one formulated the problem in terms of quantities containing terms of $(x - a_i)$ and maintained the special definitions imposed

A later work by Föppl[6] was published in 1914, in which he presented the same concept as Clebsch but introduced slightly more formalism to the method Föppl's "comma technique," as it is sometimes called, uses commas to separate the terms and keep one mindful of the definitions Föppl illustrates the method by an example identical to that of Fig 1

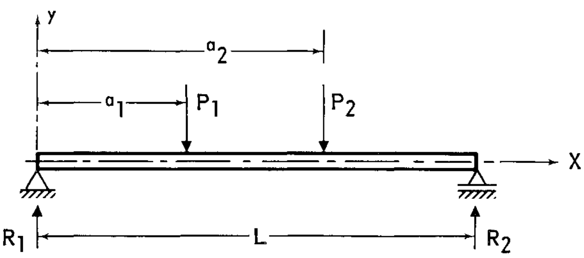

Fig 1 Simply supported beam with concentrated loads

The first English language publication was apparently that of Macauley[7] in 1919 and is sometimes referenced as the origin of the method His brief two-page article is the first attempt to define a formalism to keep track of the definitions implied by Clebsch Macauley defines a function

$$\{f(x)\}_a = \begin{cases} 0 & \text{if } x < a \\ f(x) & \text{if } x \geq a \end{cases} \quad (4)$$

in which he uses the braces and subscript to indicate that the function is zero when the argument is less than the subscript This special notation gives an air of formalism to the esoteric properties originally implied by Clebsch This is the origin of the "Macauley's brackets" mentioned by Bahar[3] and Urry,[4] although the form in which they use the brackets was later introduced by Brock and Newton[8] Macauley also extends the use of his formalism for uniform load packages, whereas Clebsch and Föppl consider only concentrated loads Macauley's formalism, however, does not maintain the similarity between his equations and those of the classical method as nicely as the form of Brock and Newton

Brown (1944)[9] approaches the same problem in terms of Heaviside's unit step function and achieves the same result in a more mathematically formal way He shows that this method was also applicable to discontinuities in EI arising in stepped beams

Conway[10] published a text in 1950 on basic mechanics of materials in which he presents the method by means of one example of a simply supported beam carrying one concentrated force His example parallels the original presentation of Clebsch but with little supporting explanation

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Brock and Newton,[8] in 1952, were apparently the first to define the present formalism of the Clebsch Method They systematically exploit its advantage, using the following special notation:

$$\{x - a\}^n = \begin{cases} 0 & \text{if } x < a \quad n = 0,1,2 \\ (x - a)^n & \text{if } x \geq a \end{cases} \tag{5}$$

and, for $n = 0$,

$$\{x - a\}^0 = \begin{cases} 0 \text{ if } x < a \\ 1 \text{ if } x > a \end{cases} \tag{6}$$

with the further stipulation that, when $x = a$ in Eq (6), the function can assume both values They use special braces, as did Macauley, to imply the special definitions but in a manner quite different

With this notation, the shear V, moment M, slope θ, and deflection y expressions for the beam of Fig 1 can be written in very compact and convenient form as

$$\begin{aligned} V &= R_1 - P_1\langle x - a_1\rangle^0 - P_2\langle x - a_2\rangle^0 \\ M &= R_1x - P_1\langle x - a_1\rangle^1 - P_2\langle x - a_2\rangle^1 \\ EI\theta &= \tfrac{1}{2}R_1x - \tfrac{1}{2}P_1\langle x - a_1\rangle^2 - \tfrac{1}{2}P_2\langle x - a_2\rangle^2 + C_1 \\ EIy &= \tfrac{1}{6}R_1x - \tfrac{1}{6}P_1\langle x - a_1\rangle^3 - \tfrac{1}{6}P_2\langle x - a_2\rangle^3 + C_1x + C_2 \end{aligned} \tag{7}$$

where the "less than-greater than" brackets $\langle\rangle$ replace the braces $\{\}$ for greater emphasis These equations are the same as those originally written by Clebsch, Eqs (1–3), plus the fact that the shear equation can also be included by making use of the zero exponent Brock and Newton include the application of their formalism to distributed load packages as a part of their systematic development For example, the beam of Fig 2a can be replaced with the statically equivalent case of Fig 2b The shear and moment equations are then

$$V = R_1 - q\langle x - a\rangle + q\langle x - b\rangle \tag{8a}$$

$$M = R_1x - \tfrac{1}{2}q\langle x - a\rangle^2 + \tfrac{1}{2}q\langle x - b\rangle^2 \tag{8b}$$

where, by the definitions, the load package once started at $x = a$ continues for all $x > a$ Consequently, a mirror load package must be subtracted where the original load package ends This parallels the use of step functions to describe a square wave

Thus, Brock and Newton, who reference Föppl and Conway as the basis for their development, adopted a special bracket notation to accentuate the special definitions patterned after Föppl and to maintain the similarity between the equations of the classical method and those of the Clebsch method Macauley, who introduced the use of special brackets, did so in a manner and form different from the concept of Clebsch and analogous to the Heaviside unit step function The Brock-Newton formulation for load packages is quite different in appearance from that of Macauley, although both formulations lead to identical results Thus, the method in its present form can be traced to Clebsch and Föppl for the concept and to Brock and Newton for the formalism

Crandall and Dahl[1] present the method in a text published in 1959 using the bracket formalism of Brock and Newton They add one supplementary definition in the use of negative exponents which permits the writing of equations for the loading including concentrated forces and moments The text by Panlilio,[2] 1963, presents the Clebsch method with the formalism of Brock and Newton using braces rather than the notation of Eq (7) Panlilio cites Föppl and Macauley as the basis for his presentation

Urry,[4] in his recent note, applied the Clebsch method to beams with axial loading in addition to the transverse loading, showing a formidable extension to the usefulness of the method This work suggests possible simplifications to be achieved in other areas by judicious application of the method

The reader who is interested in inquiring further into the use of the Clebsch method can refer to the texts by Crandall and Dahl[1] and by Panlilio[2] and to the article by Brock and Newton [8] The two texts include general discussions of singularity functions The Brock and Newton article, although not as readily available, is recommended for those interested in practical application with a minimum of sophistication

The advantages of the Clebsch method in a great majority of the beam deflection problems encountered is so tremendous that it is difficult to believe that it has not received much wider use Perhaps it has been overlooked in our preference for sophistication over simplification It is an analytical method as simple to apply as the simplifications resulting from it The Clebsch method was fully integrated into the basic Mechanics of Materials course at Washington University by G Mesmer prior to this writer's first exposure to it in 1955, and it has been continued since through supplementary class notes[11] prepared to aid the student It is easily mastered by a second-year student and brings many more deflection and statically indeterminate problems within the realm of class excercises than is practical with the classical method of solution by separate integrations But, because it is within the grasp of the beginning student, its usefulness to the practicing engineer should not be considered trivial The logic

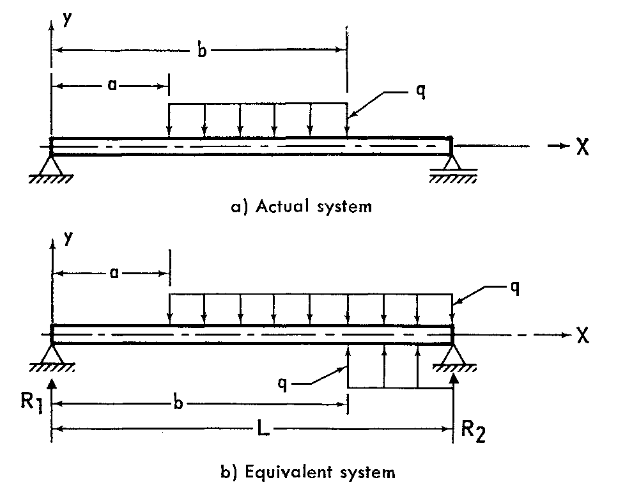

Fig 2 Simply supported beam with uniform load package

of digital computers is such that the Clebsch method is particularly well suited for use with such machines The simplest fomulations for a machine solution of the redundant reactions of a beam on N supports look almost as they did when Clebsch formulated the problem over 100 years ago

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Analysis of a Symmetrically Loaded Sandwich Cylinder

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Nomenclature

$D_x\ D_y$ = beam flexural stiffnesses per inch of width of orthotropic shell in axial and circumferential directions respectively, in lb

D_{Q_x} = shear stiffness in xz plane per inch of width, lb/in

E_x, E_y = extensional stiffnesses of orthotropic shell in axial and circumferential directions, respectively, lb/in

G_c = core shear modulus in xz plane, psi

M_x = moment acting in the x direction, in lb/in

N = tension force acting in the centroidal plane, lb/in

p = pressure acting on cylinder in direction normal to plane of sandwich, psi

u,v,w = displacement in x, y, and z directions, respectively

Q_x = transverse shear force acting in xz plane, lb/in

μ_x, μ_y = Poisson's ratio associated with bending in x and y directions, respectively

μ_x', μ_y' = Poisson's ratios associated with extension in x and y directions, respectively

Introduction

A PROCEDURE for analyzing homogeneous isotropic cylinders loaded symmetrically along the longitudinal axis was presented by Timoshenko [1] In that analysis, it was assumed that shear distortion is negligible, compared with bending distortion However, in the case of a cylinder constructed from a sandwich with a relatively low traverse shear rigidity, the shear distortion may not be negligible Therefore, an analysis is presented in the following paragraphs that includes shear distortion for a symmetrically loaded orthotropic sandwich cylinder The shear deformations are taken into consideration by the same method used by Libove and Batdorf[2] in accounting for shear deformations in a sandwich plate

Derivation of the Differential Equation

As shown in Fig 1, the cylinder is loaded symmetrically with respect to the longitudinal axis Only small deflections

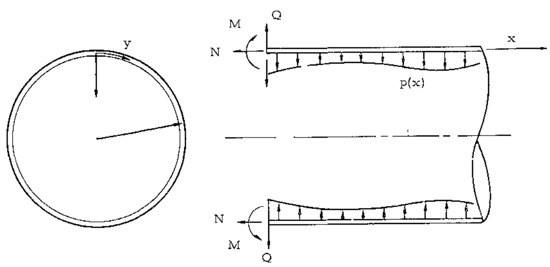

Fig 1 Cylinder subjected to symmetrical external forces along axis

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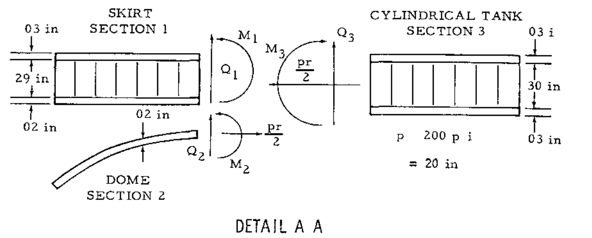

Fig 2 Pressure vessel joint detail

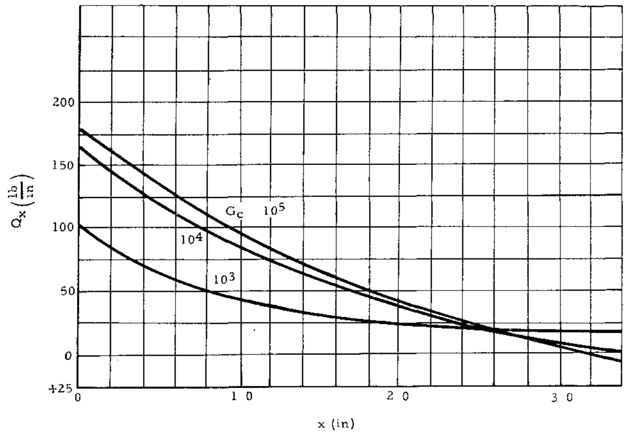

Fig 3 Effects of shear deflection on transverse shear force

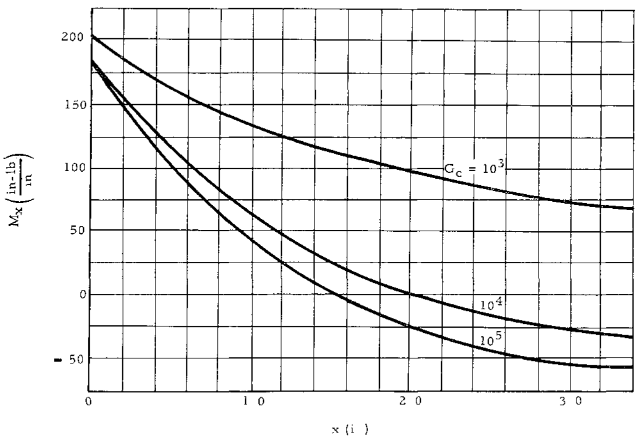

Fig 4 Effects of shear deflection on internal moments

are considered Because of symmetry, the resultant moment in the x direction, M_x, is written as a function of just two dependent variables:

$$M_x = -\frac{D_x}{1 - \mu_x\mu_y}\left[\frac{d^2w}{dx^2} - \frac{1}{D_{Q_x}}\frac{dQ_x}{dx}\right] \tag{1}$$

The three equations of equilibrium are

$$(dN_x/dx) = 0 \tag{2}$$

$$N_x\frac{d^2w}{dx^2} + \frac{dQ_x}{dx} + \frac{N_y}{r} + p = 0 \tag{3}$$

$$(dM_x/dx) - Q_x = 0 \tag{4}$$

It can be seen from Eq (2) that the force N_x is constant; therefore $N_x = N_0$

The forces in the middle surface of an orthotropic shell in terms of the deflections are obtained from Ref 3:

$$N_0 = \frac{E_x}{1 - \mu_x'\mu_y'}\left[\frac{du}{dx} + \mu_y'\left(-\frac{w}{r}\right)\right] \tag{5}$$

$$N_y = \frac{E_y}{1 - \mu_x'\mu_y'}\left[\left(-\frac{w}{r}\right) + \mu_x'\frac{du}{dx}\right] \tag{6}$$